The Theory of Integer Partitions, I

Trine Mathematics Colloquium

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The Big Questions

- What is a partition?
- What tools do we have to work with?
- What do these tools tell us?

Combinatorics

Combinatorics is the mathematical discipline of counting abstract objects.

A **partition** λ is a finite non-increasing sequence of positive integers. The **weight** of λ is the sum of its integer parts.

$$(4,3,2,1) \qquad (1,1,1) \qquad () \qquad (1000,999,17,4)$$

The relation $\lambda \vdash n$ means "the weight of λ is n". We also say that "λ is a partition of n".

Counting by List

Let $p(n)$ denote the number of partitions of n. How do we calculate this number? How quickly does it grow?

Let's enumerate all partitions of 4. These are

$$\begin{array}{ll} (4) & (3,1) \\ (2,2) & (2,1,1) \\ (1,1,1,1). & \end{array}$$

Therefore, $p(4) = 5$.

Here are some values of $p(n)$ for small n.

$p(0) = 1$	$p(5) = 7$	$p(10) = 42$	$p(15) = 176$
$p(1) = 1$	$p(6) = 11$	$p(11) = 56$	$p(16) = 231$
$p(2) = 2$	$p(7) = 15$	$p(12) = 77$	$p(17) = 297$
$p(3) = 3$	$p(8) = 22$	$p(13) = 101$	$p(18) = 285$
$p(4) = 5$	$p(9) = 30$	$p(14) = 135$	$p(19) = 490$

Percy MacMahon (1854 - 1929) was renowned for his work in enumerating partitions by hand. He developed many generalizations of partition theory.

Counting By Picture

Lemma

For all $n \geq 0$, we have that $p(n) \leq 2^{n-1}$.

Proof

Consider n dots in a row:

$$\circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ \quad \circ$$

There are $n-1$ gaps between the dots, where we may choose to insert a plus sign, giving 2^{n-1} possible arrangements of n dots and up to $(n-1)$ plus signs.

$$\circ + \circ \circ \circ + \circ \circ + \circ$$

In such an arrangement, replace each cluster of j dots by the integer j.

$$\circ + \circ\circ\circ + \circ\circ + \circ \quad \mapsto \quad 1+3+2+1$$

Then, arrange these integer parts into non-increasing order to produce a partition whose weight is n.

$$3+2+1+1 \quad \mapsto \quad (3,2,1,1)$$

All partitions $\lambda \vdash n$ are formed this way, albeit non-uniquely. Therefore, $p(n) \leq 2^{n-1}$. □

Leonard Euler (1707 - 1783) formulated many partition identities by defining bijections between sets of partitions, and also by manipulation of generating functions.

Counting by Bijection

Theorem (Euler)

For all $n \geq 0$, the number of partitions $\lambda \vdash n$ with distinct parts is equal to the number of partitions $\mu \vdash n$ with only odd parts.

Proof

Let λ be a partition with distinct parts such that $\lambda \vdash n$.
Perform the following algorithm on λ:

1. If λ contains an even part j, replace j by two parts of size $j/2$.
2. Repeat (1) until λ only consists of odd parts.

For example, let $\lambda = (6, 4, 3, 2)$

$$\begin{gathered}(6,4,3,2)\\(6,4,3,1,1)\\(4,3,3,3,1,1)\\(3,3,3,2,2,1,1)\\(3,3,3,2,1,1,1,1)\\(3,3,3,1,1,1,1,1,1)\end{gathered}$$

The corresponding partition is $\mu = (3, 3, 3, 1, 1, 1, 1, 1, 1)$.

Each λ corresponds to exactly one μ according to this map, because the map is invertible. A bijection between two finite sets demonstrates that they have the same size. □

Corollary (Non-partition version)

Binary representation of non-negative integers is unique.

$$
\begin{aligned}
0 &= 0_{(2)} \\
1 &= 1_{(2)} \\
2 &= 10_{(2)} \\
3 &= 11_{(2)} \\
4 &= 100_{(2)} \\
&\vdots
\end{aligned}
$$

What's that got to do with partitions?

Corollary (Partition version)

For all $n \geq 0$, there is a unique partition $\lambda \vdash n$ consisting only of parts which are distinct powers of two.

Proof

Given n, there is a unique partition $\mu = (1, 1, \ldots 1) \vdash n$. Note that μ consists solely of odd parts. Perform the following algorithm on μ:

1. If μ contains two equal parts of size j, replace the pair by a single part of size $2j$.
2. Repeat (1) until μ only consists of distinct parts.

This produces the unique partition λ as desired. □

Counting by Generating Function

Let $a(n)$ be a sequence defined for $n \geq 0$. The generating function of $a(n)$ is the series

$$A(q) = \sum_{n=0}^{\infty} a(n)q^n.$$

Here, q is an indeterminate. These series may be manipulated in the ring of formal power series $\mathbb{C}[[q]]$.

(That is, $+$ and $\times$ work as expected.)

Theorem (Euler Product)

Let

$$P(q) = \sum_{n \geq 0} p(n) q^n = 1 + q + 2q^2 + 3q^3 + 5q^4 + \cdots .$$

Then,

$$P(q) = \prod_{i=1}^{\infty} \frac{1}{1 - q^i}.$$

Proof

Expand each term of the Euler product using the geometric series formula to obtain

$$\prod_{i=1}^{\infty} \frac{1}{1-q^i} = \prod_{i=1}^{\infty} (1 + q^i + q^{2i} + q^{3i} + \cdots)$$
$$= \prod_{i=1}^{\infty} (1 + q^i + q^{i+i} + q^{i+i+i} + \cdots).$$

For all $n \geq 0$, the monomial q^n appears once in the product for each way of writing

$$q^n = q^{i_1 + i_1 + \cdots + i_1} \times q^{i_2 + i_2 + \cdots + i_2} \times \cdots \times q^{i_k + i_k + \cdots + i_k},$$

where the i_j are distinct integers appearing in decreasing order.

Each of these representations corresponds to a unique partition $\lambda = (i_1, \ldots, i_1, i_2, \ldots, i_2, \ldots, i_k, \ldots, i_k) \vdash n$. Therefore,

$$\prod_{i=1}^{\infty} \frac{1}{1-q^i} = \sum_{n=0}^{\infty} \left(\sum_{\lambda \vdash n} q^n \right) = \sum_{n=0}^{\infty} p(n) q^n.$$

□

Corollary

Let $d(n)$ *be the number of partitions* $\lambda \vdash n$ *such that* λ *consists of distinct parts. Then,*

$$\sum_{n=0}^{\infty} d(n)q^n = \prod_{i=1}^{\infty} 1 + q^i.$$

Corollary

Let $o(n)$ *be the number of partitions* $\lambda \vdash n$ *such that* λ *consists of odd parts. Then,*

$$\sum_{n=0}^{\infty} o(n)q^n = \prod_{i=1}^{\infty} \frac{1}{1 - q^{2i-1}}.$$

Theorem (Euler)

For all $n \geq 0$, we have that $d(n) = o(n)$.

Proof

$$\prod_{i=1}^{\infty} 1+q^i = \prod_{i=1}^{\infty} \frac{(1+q^i)(1-q^i)}{1-q^i} = \frac{\prod_{i=1}^{\infty} 1-q^{2i}}{\prod_{i=1}^{\infty} 1-q^i} = \prod_{i=1}^{\infty} \frac{1}{1-q^{2i-1}}.$$

Therefore,

$$\sum_{n=0}^{\infty} d(n)q^n = \sum_{n=0}^{\infty} o(n)q^n.$$

Because series are uniquely determined by their coefficients, the two coefficient sequences must be identical. □

Godfrey Harold Hardy (1877 - 1947) studied partitions for their relationship to complex analysis.

Srinivasa Ramanujan (1887 - 1920) was a self-taught mathematician who had a profound intuition for partition theory.

Counting by Complex Analysis

More advanced results in partition theory follow from the treatment of generating functions as analytic series in the complex plane.

Theorem (Hardy, Ramanujan)

Let $p(n)$ denote the number of partitions of n. Then,

$$p(n) \sim \frac{\exp(\pi\sqrt{2n/3})}{4n\sqrt{3}}.$$

Here, $a(n) \sim b(n)$ denotes the relation

$$\lim_{n\to\infty} \frac{a(n)}{b(n)} = 1.$$

Ramanujan discovered a pattern in $p(n)$ by observing a table like the one we saw earlier.

$p(0) = 1$	$p(5) = 7$	$p(10) = 42$	$p(15) = 176$
$p(1) = 1$	$p(6) = 11$	$p(11) = 56$	$p(16) = 231$
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Note that each $p(n)$ on the bottom row is a multiple of five.

Do you expect this pattern to continue? Do you expect this pattern to work for a table with more than five rows?

Theorem (The Ramanujan Congruences)

For all $n \geq 0$,

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5} \\ p(7n+5) &\equiv 0 \pmod{7} \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

Since the time of Ramanujan, it has been discovered that $p(n)$ exhibits other, more complicated congruences for all moduli coprime to 6.

Proof Sketch

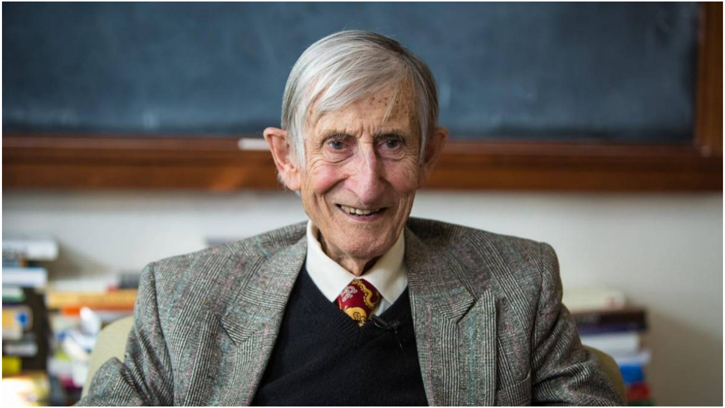
Consider the generating function for $p(5n+4)$,

$$\sum_{n=0}^{\infty} p(5n+4)q^n.$$

It can be shown that

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q)_\infty^5}{(q;q)_\infty^6} = 5\sum_{n=0}^{\infty} a(n)q^n = \sum_{n=0}^{\infty} 5a(n)q^n.$$

Therefore, each $p(5n+4)$ is equal to a multiple of five. The other two congruences are proved similarly. $\square$



Freeman Dyson (1923 - 2020) was interested in finding a simpler proof of the Ramanujan congruences by measuring statistics of individual partitions.

Counting by Rank

Let λ be a partition. The **rank** of λ is an integer equal to the largest part of λ minus the number of parts which occur in λ.

$$r(4,2,1) = 4 - 3 = 1.$$

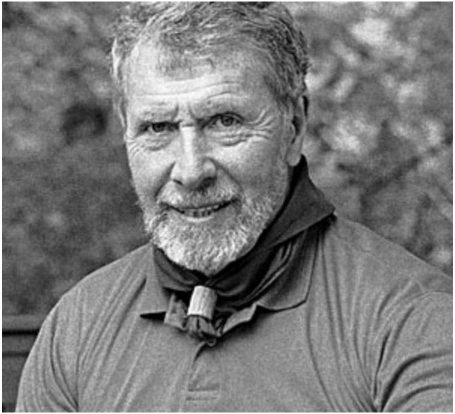
Dyson observed that the partitions $\lambda \vdash (5n+4)$ may be grouped into subsets of equal size according to their rank modulo five. He conjectured that this grouping would lead to a combinatoric proof of the first two Ramanujan Congruences.

Here are the partitions $\lambda \vdash 4$.

λ	$r(\lambda)$	$r(\lambda) \pmod 5$
(4)	3	3
$(3,1)$	1	1
$(2,2)$	0	0
$(2,1,1)$	-1	4
$(1,1,1,1)$	-3	2

Note that each of the residues 0, 1, 2, 3, and 4 occur the same number of times in the last column.

You might also notice something interesting in the middle column.



Arthur Oliver Lonsdale Atkin (1925 - 2008) was a proponent of using computer assisted calculation to further knowledge of complex analysis.

Peter Swinnerton-Dyer (1927 - 2018) was known for his foundational work in the theory of elliptic curves and L-functions, both belonging to the field of complex analysis.

Atkin and Swinnerton-Dyer proved Dyson's conjecture true in 1954: the size of the sets

$$\{\lambda \vdash (5n+4) : r(\lambda) \equiv i \mod 5\}$$

do not depend on the residue i. Therefore, all the partitions of $5n+4$ may be placed into five bins of equal size, which demonstrates that $p(5n+4) \equiv 0$ modulo 5.

The same two mathematicians proved a similar result connecting ranks and the modulo 7 congruence.

Dyson was aware that the rank method fails to establish Ramanujan's modulo 11 congruence. (Try it yourself on the partitions of 6 and see what goes wrong!)

He further conjectured that another function would fill this gap. Dyson named this function the **crank** of a partition, but did not supply a formula for it.

A working definition of the crank function was later established by Frank Garvan and George Andrews.

George Andrews (1938 -) has been a leading figure in the theory of partitions through the 20th century and continues to publish today.

Frank Garvan (1955 -) is another leading figure in the theory of partitions. He maintains a software suite for computer calculation of generating series at `qseries.org`.

Counting by Crank

Let λ be a partition. If λ does not contain any parts of size 1, then the **crank** of λ is defined to be equal to the largest part of λ.

Otherwise, define $w(\lambda) > 0$ to be the number of parts of size 1 which occur in λ. Then define $m(\lambda)$ to be the number of parts of λ which are greater than $w(\lambda)$ in size. In this case, $c(\lambda) = m(\lambda) - w(\lambda)$.

$$c(5,4,2) = 5$$
$$c(5,4,1) = 2 - 1 = 1$$

This strange looking formula comes from starting with the series

$$\prod_{i=1}^{\infty} \frac{1-q^i}{(1-zq^i)(1-q^i/z)}$$

and reverse-engineering what its coefficients are counting.

Theorem (Andrews, Garvan)

For all $n \geq 0$, the size of each of the sets

$$\{\lambda \vdash (5n+4) : c(\lambda) \equiv i \mod 5\}$$
$$\{\lambda \vdash (6n+5) : c(\lambda) \equiv i \mod 7\}$$
$$\{\lambda \vdash (11n+6) : c(\lambda) \equiv i \mod 11\}$$

does not depend on the choice of residue i. Therefore, enumeration via the crank function is sufficient to establish all three Ramanujan congruences.

Conclusion

This talk represents only a tiny portion of the past 300 years of research into integer partitions. Please join me again in the near future when I discuss more of these results, including my own contributions to the field.

Thank You!